

ON A PROBLEM OF SIDON FOR POLYNOMIALS OVER FINITE FIELDS

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ABSTRACT. Let ω be a sequence of positive integers. Given a positive integer n , we define

$$r_n(\omega) = |\{(a, b) \in \mathbb{N} \times \mathbb{N} : a, b \in \omega, a + b = n, 0 < a < b\}|.$$

S. Sidon conjectured that there exists a sequence ω such that $r_n(\omega) > 0$ for all n sufficiently large and, for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{r_n(\omega)}{n^\epsilon} = 0.$$

P. Erdős proved this conjecture by showing the existence of a sequence ω of positive integers such that

$$\log n \ll r_n(\omega) \ll \log n.$$

In this paper, we prove an analogue of this conjecture in $\mathbb{F}_q[T]$, where \mathbb{F}_q is a finite field of q elements. More precisely, let ω be a sequence in $\mathbb{F}_q[T]$. Given a polynomial $h \in \mathbb{F}_q[T]$, we define

$$r_h(\omega) = |\{(f, g) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T] : f, g \in \omega, f + g = h, \deg f, \deg g \leq \deg h, f \neq g\}|.$$

We show that there exists a sequence ω of polynomials in $\mathbb{F}_q[T]$ such that

$$\deg h \ll r_h(\omega) \ll \deg h$$

for $\deg h$ sufficiently large.

1. INTRODUCTION

In the course of investigations on Fourier series by S. Sidon, several questions arose concerning the existence and nature of certain positive integer sequences ω for which $r_n(\omega) = |\{(a, b) \in \mathbb{N} \times \mathbb{N} : a, b \in \omega, a + b = n, 0 < a < b\}|$ is bounded or, in some sense, exceptionally small, where $|S|$ denotes the cardinality of the set S . In particular, he asked the following question in 1932, known as the Sidon Problem [1]:

Does there exist a sequence ω such that $r_n(\omega) > 0$ for all n sufficiently large and, for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{r_n(\omega)}{n^\epsilon} = 0 ?$$

In 1954, P. Erdős answered positively to the question by proving the following [1]:

Theorem (Erdős). *There exists a sequence ω such that*

$$\log n \ll r_n(\omega) \ll \log n$$

for all n sufficiently large.

In other words, there exists a “thin” set ω such that every positive integer sufficiently large can be represented as a sum of two elements in ω . On the other direction, Erdős and Rényi proved the following theorem in [2] that there exists a “thick” set ω such that $r_n(\omega)$ is bounded for all n .

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Theorem (Erdős-Rényi). *For any $\varepsilon > 0$, there exists a positive number $G = G(\varepsilon)$ and a sequence ω , such that $r_n(\omega) < G$ for all n and*

$$|\{m \in \omega : m \leq n\}| > n^{\frac{1}{2}-\varepsilon}$$

for sufficiently large n .

We note that the result is best possible up to the ε term. One way to see this fact is by the pigeon hole principle. Suppose we have $\omega_0 \subseteq \mathbb{N}$, where $r_n(\omega_0) < G$ for all $n \in \mathbb{N}$. Given any $m_1, m_2 \in \{m \in \omega_0 : m \leq n\}$, we have $1 < m_1 + m_2 \leq 2n$. Therefore, by the pigeon hold principle, it follows that

$$G > \max_{1 < m \leq 2n} r_m(\omega_0) \geq \frac{|\{m \in \omega_0 : m \leq n\}|^2 - |\{m \in \omega_0 : m \leq n\}|}{2(2n-1)}.$$

Consequently, we obtain

$$|\{m \in \omega_0 : m \leq n\}| \ll n^{1/2}.$$

In this paper, we prove an analogue of these results in the setting of $\mathbb{F}_q[T]$.

Let ω be a sequence of polynomials in $\mathbb{F}_q[T]$. For each $h \in \mathbb{F}_q[T]$, we define

$$r_h(\omega) = |\{(f, g) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T] : f, g \in \omega, h = f + g, \deg f, \deg g \leq \deg h, f \neq g\}|.$$

Note $\deg f$ is the degree of $f \in \mathbb{F}_q[T]$ with the convention that $\deg 0 = -\infty$. We prove the following results.

Theorem 1. *There exists a sequence ω of polynomials in $\mathbb{F}_q[T]$ such that*

$$\deg h \ll r_h(\omega) \ll \deg h$$

for $\deg h$ sufficiently large.

On the other direction, we prove that there exists a “thick” set with bounded value $r_h(\omega)$. We denote the elements of ω by $\omega = \{f_i\}_{i \in \mathbb{N}}$, where $\deg f_i \leq \deg f_j$ ($i < j$).

Theorem 2. *For each $\epsilon > 0$, there exists a sequence $\omega = \{f_i\}$ of polynomials in $\mathbb{F}_q[T]$ and a positive integer K such that $r_h(\omega) < K$ for all $h \in \mathbb{F}_q[T]$ and $q^{\deg f_i} \ll i^{2+\epsilon}$.*

For each $h \in \mathbb{F}_q[T]$, we define

$$t_h(\omega) = |\{(f, g) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T] : f, g \in \omega, h = f - g, \deg f, \deg g \leq \deg h\}|.$$

We also prove the following variation of the existence of thick sets.

Theorem 3. *For each $\epsilon > 0$, there exists a sequence $\omega = \{f_i\}$ of polynomials in $\mathbb{F}_q[T]$ and a positive integer K' such that $t_h(\omega) < K'$ for all $h \in \mathbb{F}_q[T]$ and $q^{\deg f_i} \ll i^{2+\epsilon}$.*

We prove our theorems following the methods of Chapter III of [3], which utilizes the language of probability. Roughly speaking, we set up a probability space to study the probability of the events $\{\omega | r_h(\omega) = d\}$ for all non-negative integer d . Using the Borel-Cantelli lemma, we show that the sequences satisfy the desired properties with probability 1. We also remark that Theorems 2 and 3 have been generalized to m -fold sums and differences by K. E. Hare and the second author in [4].

The organization of this paper is as follows. In Section 2, we first review the basic probability theory and state the Borel-Cantelli lemma. Next, in Section 3, we state the equivalent statements of our theorems and set up the probability space used in our proof. In Section 4, we establish several technical lemmas. Finally, the remaining sections are devoted to the proof of our main results.

2. PRELIMINARIES

We start with probability theory. Let $\{X_j\}$ be a sequence of spaces and write

$$X = \prod_{j=0}^{\infty} X_j.$$

Let \mathcal{M}_j be a σ -algebra of subsets of X_j . A measurable rectangle with respect to the sequence $\{\mathcal{M}_j\}$ is defined to be a subset W of X which is representable in the form

$$W = \prod_{j=0}^{\infty} W_j,$$

where $W_j \in \mathcal{M}_j$ and $W_j = X_j$ except for finitely many j . The following two theorems are standard results in probability theory, see for example [3, p. 123, Thm. 5] and [3, p. 135] for reference.

Theorem 4. [3, p. 123, Theorem 5] *Let $\{(X_j, \mathcal{M}_j, P_j)\}_{j \geq 0}$ be a sequence of probability spaces, and write*

$$X = \prod_{j=0}^{\infty} X_j.$$

Let \mathcal{M} be the minimal σ -algebra of subsets of X containing every measurable rectangle with respect to the sequence $\{\mathcal{M}_j\}$. Then there exists a unique measure P on \mathcal{M} with the property that for every non-empty measurable rectangle W ,

$$(5) \quad P(W) = \prod_{j=0}^{\infty} P_j(W_j),$$

where the W_j are defined by $W = \prod_{j=0}^{\infty} W_j$, $W_j \in \mathcal{M}_j$ ($j \geq 0$).

We remark that the product in (5) is, in essence, a finite product by the definition of measurable rectangles with respect to the sequence $\{\mathcal{M}_j\}$. Furthermore, since

$$P(X) = \prod_{j=0}^{\infty} P_j(X_j) = 1,$$

the σ -algebra \mathcal{M} in conjunction with the measure P constitutes a probability space (X, \mathcal{M}, P) .

Theorem 6. [3, p. 135, The Borel-Cantelli Lemma] *Let (X', \mathcal{M}', P') be a probability space. Let $\{W_\ell\}$ be a sequence of measurable events. If*

$$\sum_{\ell=1}^{\infty} P'(W_\ell) < \infty,$$

then, with probability 1, at most finite number of the events W_ℓ can occur; or, equivalently,

$$P' \left(\bigcap_{i=1}^{\infty} \bigcup_{\ell=i}^{\infty} W_\ell \right) = 0.$$

3. PROBABILITY SPACE (Ω, \mathcal{M}, P)

We let $q = p^s$ for a prime p , and denote \mathbb{F}_q to be the finite field of q elements. Let $\mathbb{F}_q[T]$ be the polynomial ring over \mathbb{F}_q . Let ι be any bijective map from $\mathbb{Z} \cap [0, q-1]$ to \mathbb{F}_q . We label each of the polynomials in $\mathbb{F}_q[T]$ as follows. Let $\mathbb{Z}_{\geq 0}$ be the set of all non-negative integers. For every $N \in \mathbb{Z}_{\geq 0}$, we define

$$p_N := \iota(c_0) + \iota(c_1)T + \dots + \iota(c_n)T^n,$$

where $N = c_0 + c_1q + \dots + c_nq^n$ and $0 \leq c_i < q$ ($1 \leq i \leq n$). It is clear that this identification gives a one-to-one correspondence of sets between $\mathbb{Z}_{\geq 0}$ and $\mathbb{F}_q[T]$.

We use ω to denote a subsequence of the sequence of all polynomials in $\mathbb{F}_q[T]$, i.e. $p_0, p_1, p_2, p_3, \dots$ and Ω to denote the space of all such sequences ω . By $f \in \omega$, we mean $f \in \mathbb{F}_q[T]$ appears in the sequence ω . Given $N \in \mathbb{Z}_{\geq 0}$ and $\omega \in \Omega$, we define

$$r_N(\omega) = |\{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : p_a, p_b \in \omega, p_N = p_a + p_b, \deg p_a, \deg p_b \leq \deg p_N, a < b\}|,$$

and

$$t_N(\omega) = |\{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} : p_a, p_b \in \omega, p_N = p_a - p_b, \deg p_a, \deg p_b \leq \deg p_N\}|.$$

We prove the following results which our main theorems, namely Theorems 1, 2 and 3, are consequences of.

Theorem 7. *There exists a sequence ω of polynomials in $\mathbb{F}_q[T]$ such that*

$$\log N \ll r_N(\omega) \ll \log N$$

for N sufficiently large.

Theorem 8. *For each $\epsilon > 0$, there exists a sequence $\omega = \{p_{b_j}\}$ of polynomials in $\mathbb{F}_q[T]$ and a positive integer K_0 such that $r_N(\omega) < K_0$ for all $N \in \mathbb{Z}_{\geq 0}$ and $b_j \ll j^{2+\epsilon}$.*

Theorem 9. *For each $\epsilon > 0$, there exists a sequence $\omega = \{p_{b_j}\}$ of polynomials in $\mathbb{F}_q[T]$ and a positive integer K'_0 such that $t_N(\omega) < K'_0$ for all $N \in \mathbb{Z}_{\geq 0}$ and $b_j \ll j^{2+\epsilon}$.*

Since $\deg p_N \leq \log_q N < \deg p_N + 1$, we can easily derive Theorems 1, 2, and 3, from Theorems 7, 8, and 9, respectively.

We now prove the existence of the following probability space. The content of this theorem is essentially [3, p. 141, Theorem 13].

Theorem 10. *Let*

$$\alpha_0, \alpha_1, \alpha_2, \alpha_3, \dots,$$

be real numbers satisfying $0 \leq \alpha_i \leq 1$ ($i \geq 0$). Then there exists a probability space (Ω, \mathcal{M}, P) with the following two properties:

- (i) *For every non-negative integer m , the event $\mathfrak{B}_m = \{\omega \in \Omega : p_m \in \omega\}$ is measurable and $P(\mathfrak{B}_m) = \alpha_m$.*
- (ii) *The events $\mathfrak{B}_0, \mathfrak{B}_1, \mathfrak{B}_2, \dots$ are independent.*

PROOF. Let Y be the space of two elements, y_0 and y_1 say. For each sequence ω we associate the sequence $\{x_j\}$ of elements of Y , defined by

$$x_j = \begin{cases} y_0, & \text{if } p_j \notin \omega, \\ y_1, & \text{if } p_j \in \omega, \end{cases}$$

for $j \geq 0$. The space X consisting of all the sequences $x = \{x_j\}$ is given by

$$X = \prod_{j=0}^{\infty} X_j,$$

where $X_j = Y$ for $j \geq 0$. Let $\mathcal{M}_j = \{\phi, \{y_0\}, \{y_1\}, X_j\}$, the non-trivial σ -algebra of X_j , and let P_j be the probability measure on \mathcal{M}_j such that $P_j(\{y_1\}) = \alpha_j$.

We apply Theorem 4 to the sequence $\{X_j, \mathcal{M}_j, P_j\}$ of probability spaces. In view of the one-to-one correspondence between the elements of X and Ω , we may denote the resulting probability space as (Ω, \mathcal{M}, P) .

Now, we prove (Ω, \mathcal{M}, P) satisfies the two properties (i) and (ii). Clearly, we have

$$\mathfrak{B}_m = \{\omega \in \Omega : p_m \in \omega\} = \prod_{j=0}^{\infty} W_j,$$

where $W_j = X_j$ for all j except $j = m$ and $W_m = \{y_1\}$. Then, (i) follows, because $\mathfrak{B}_m \in \mathcal{M}$ by the definition of \mathcal{M} , and by (5) we have

$$P(\mathfrak{B}_m) = \prod_{j=0}^{\infty} P_j(W_j) = P_m(\{y_1\}) = \alpha_m.$$

For (ii), we consider any finite subset of $\{\mathfrak{B}_j\}$, say $\mathfrak{B}_{j_1}, \mathfrak{B}_{j_2}, \dots, \mathfrak{B}_{j_\ell}$. Then, clearly we have

$$\bigcap_{i=1}^{\ell} \mathfrak{B}_{j_i} = \{\omega \in \Omega : p_{j_i} \in \omega \ (1 \leq i \leq \ell)\} = \prod_{j=0}^{\infty} W_j,$$

where $W_j = X_j$ for all j except $j = j_1, \dots, j_\ell$ and $W_{j_i} = \{y_1\}$ for $1 \leq i \leq \ell$. Thus, by (5) and (i) we obtain

$$P\left(\bigcap_{i=1}^{\ell} \mathfrak{B}_{j_i}\right) = \prod_{j=0}^{\infty} P_j(W_j) = \prod_{i=1}^{\ell} P_{j_i}(\{y_1\}) = \prod_{i=1}^{\ell} \alpha_{j_i} = \prod_{i=1}^{\ell} P(\mathfrak{B}_{j_i}),$$

from which (ii) follows. ■

4. TECHNICAL LEMMAS

In this section, we prove several technical lemmas used in our proofs. For each $N \in \mathbb{Z}_{\geq 0}$, let $p_N \in \mathbb{F}_q[T]$ be as prescribed in the previous section. Define

$$n := n(N) = \deg p_N = \lfloor \log_q N \rfloor.$$

Suppose $p \neq 2$. Since $\mathbb{F}_q = 2\mathbb{F}_q$, we know there exists p_{N_0} such that $p_N = p_{N_0} + p_{N_0}$. It is clear that $\deg p_{N_0} = n$; therefore, $q^n \leq N_0 < q^{n+1}$. Since $\mathbb{F}_q[T]$ is closed under addition, we can uniquely pair up the rest of polynomials of degree less than or equal to n by

$$p_N = p_a + p_{\tilde{a}},$$

where $a, \tilde{a} \in \mathbb{Z}_{\geq 0}$, $a < \tilde{a}$. We collect all such pairs (a, \tilde{a}) and form

$$A_N = \{a \in \mathbb{Z}_{\geq 0} : p_N = p_a + p_{\tilde{a}}, a < \tilde{a}, \text{ and } \deg p_a, \deg p_{\tilde{a}} \leq n\},$$

and

$$\tilde{A}_N = \{\tilde{a} \in \mathbb{Z}_{\geq 0} : p_N = p_a + p_{\tilde{a}}, a < \tilde{a}, \text{ and } \deg p_a, \deg p_{\tilde{a}} \leq n\}.$$

We have $|A_N| = |\tilde{A}_N| = (q^{n+1} - 1)/2$, and

$$\{0, \dots, q^{n+1} - 1\} = A_N \cup \tilde{A}_N \cup \{N_0\},$$

where all the unions are disjoint. Further, it is easy to see that $\{0, 1, \dots, q^n - 1\} \subseteq A_N$, because if $0 \leq a < q^n$, then p_a has degree at most $n - 1$. Thus, the corresponding $p_{\tilde{a}}$ must have degree n ; therefore, $q^n \leq \tilde{a} < q^{n+1}$. Hence, it follows that

$$(11) \quad \tilde{A}_N \subseteq \{q^n, q^n + 1, \dots, q^{n+1} - 1\}.$$

Let $M := M(N) = (q^{n+1} - 1)/2$. For convenience we label the M elements of A_N by a_i , where $1 \leq i \leq M$, and the corresponding elements of \tilde{A}_N by \tilde{a}_i .

We also define λ_N and λ'_N to be

$$\lambda_N = \sum_{1 \leq i \leq M} \alpha_{a_i} \alpha_{\tilde{a}_i},$$

and

$$\lambda'_N = \sum_{1 \leq i \leq M} \frac{\alpha_{a_i} \alpha_{\tilde{a}_i}}{1 - \alpha_{a_i} \alpha_{\tilde{a}_i}}.$$

Note when $p = 2$, for $N > 0$, we do not have to consider the polynomial p_{N_0} as above. Thus we let $M := M(N) = q^{n+1}/2$ and we can argue in a similar manner.

Define

$$s_N^*(\omega) = \sum_{m=0}^N \mathbf{1}_{\mathfrak{B}_m}(\omega),$$

where $\mathbf{1}_{\mathfrak{B}_m}$ is the characteristic function on the set \mathfrak{B}_m . Let $E(f)$ denote the expectation of a random variable f , defined by $E(f) = \int_X f \, dP$. We define

$$m_N^* = E(s_N^*) = \sum_{m=0}^N \alpha_m.$$

We need our sequence $\{\alpha_j\}$ to satisfy the following condition.

Hypothesis A. *The sequence $\{\alpha_j\}$ of probabilities (introduced in Theorem 10) satisfies the conditions: $0 < \alpha_j < 1$ ($j \geq 0$), $\{\alpha_j\}$ is monotonic and decreasing from some point onward (i.e. for $j \geq j_1$), and $\alpha_j \rightarrow 0$ as $j \rightarrow \infty$.*

We have the following result for $s_N^*(\omega)$ and its expected value m_N^* .

Lemma 12. *If, in addition to Hypothesis A,*

$$(13) \quad m_N^* \rightarrow \infty$$

as $N \rightarrow \infty$, and

$$(14) \quad \sum_{N=0}^{\infty} \frac{\alpha_N}{(m_N^*)^2} < \infty,$$

then with probability 1, we have $s_N^(\omega) \sim m_N^*$ as $N \rightarrow \infty$.*

PROOF. We denote $D^2(f)$ to be the variance of a random variable f , defined by

$$D^2(f) = E((f - E(f))^2).$$

The proof is basically an application of a variant of the strong law of large numbers [3, p. 140, Theorem 11], which is as follows. Let $\{f_j\}$ be a sequence of independent random variables, and let

$$s_i(\omega) = \sum_{j=0}^i f_j(\omega) \quad (i \geq 0).$$

Suppose we have

$$\begin{aligned} E(f_j) &> 0 \quad (j \geq 0), \\ \lim_{i \rightarrow \infty} E(s_i) &= \infty, \end{aligned}$$

and

$$\sum_{i=0}^{\infty} \frac{D^2(f_i)}{(E(s_i))^2} < \infty.$$

Then, with probability 1, we have

$$s_i(\omega) = (1 + o(1)) E(s_i)$$

as $i \rightarrow \infty$. We know that the sets \mathfrak{B}_j are independent, which is equivalent to $\mathbf{1}_{\mathfrak{B}_j}(\omega)$ being independent. Thus we apply this theorem with $f_j(\omega) = \mathbf{1}_{\mathfrak{B}_j}(\omega)$, and obtain our result. ■

For every $N, d \in \mathbb{Z}_{\geq 0}$, we define the event $\mathfrak{e}(N, d)$ as

$$\mathfrak{e}(N, d) = \{\omega \in \Omega : r_N(\omega) = d\}.$$

As mentioned in Section 1, we need to study the probability of the event $\mathfrak{e}(N, d)$. We start with the following lemma.

Lemma 15. *For all non-negative integers N and d , we have*

$$(16) \quad P(\mathfrak{e}(N, d)) = \left(\prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \right) \tilde{\sigma}_d(N),$$

where $\tilde{\sigma}_0(N) = 1$ and, if $d \geq 1$,

$$(17) \quad \tilde{\sigma}_d(N) = \sum_{1 \leq k_1 < \dots < k_d \leq M} \prod_{1 \leq i \leq d} \frac{\alpha_{a_{k_i}} \alpha_{\tilde{a}_{k_i}}}{1 - \alpha_{a_{k_i}} \alpha_{\tilde{a}_{k_i}}}.$$

PROOF. We begin with the case $d = 0$. It is easy to see that

$$\mathfrak{e}(N, 0) = \bigcap_{1 \leq k \leq M} (\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k})^c,$$

where \mathbf{c} denotes taking the complement of the set. Since the sets \mathfrak{B}_j ($j \geq 0$) are independent, we know that $\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k}$ ($1 \leq k \leq M$) are independent as $\{a_k : 1 \leq k \leq M\} \cap \{\tilde{a}_k : 1 \leq k \leq M\} = \emptyset$. Thus, it follows that $(\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k})^c$ ($1 \leq k \leq M$) are also independent. Hence, we have

$$P(\mathfrak{e}(N, 0)) = \prod_{1 \leq k \leq M} P((\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k})^c) = \prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}).$$

Suppose $1 \leq d \leq M$ and $\omega' \in \mathfrak{e}(N, d)$. Then there exist k_1, k_2, \dots, k_d such that $1 \leq k_i \leq M$, $a_{k_i}, \tilde{a}_{k_i} \in \omega'$ ($1 \leq i \leq d$), and further, if $k \neq k_i$ and $1 \leq k \leq M$, then we have either $a_k \notin \omega'$ or $\tilde{a}_k \notin \omega'$. From this observation, we can deduce that

$$P(\mathfrak{e}(N, d)) = \sum_{1 \leq k_1 < \dots < k_d \leq M} P(\mathfrak{E}(k_1, \dots, k_d)),$$

where $\mathfrak{E}(k_1, \dots, k_d)$ is the event

$$\bigcap_{1 \leq i \leq d} (\mathfrak{B}_{a_{k_i}} \cap \mathfrak{B}_{\tilde{a}_{k_i}}) \cap \bigcap_{\substack{1 \leq k \leq M \\ k \neq k_i (1 \leq i \leq d)}} (\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k})^c.$$

Again, by independence, we have

$$\begin{aligned} P(\mathfrak{E}(k_1, \dots, k_d)) &= \prod_{1 \leq i \leq d} P(\mathfrak{B}_{a_{k_i}} \cap \mathfrak{B}_{\tilde{a}_{k_i}}) \cdot \prod_{\substack{1 \leq k \leq M \\ k \neq k_i (1 \leq i \leq d)}} P((\mathfrak{B}_{a_k} \cap \mathfrak{B}_{\tilde{a}_k})^c) \\ &= \prod_{1 \leq i \leq d} \alpha_{a_{k_i}} \alpha_{\tilde{a}_{k_i}} \cdot \prod_{\substack{1 \leq k \leq M \\ k \neq k_i (1 \leq i \leq d)}} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \\ &= \prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \cdot \prod_{1 \leq i \leq d} \frac{\alpha_{a_{k_i}} \alpha_{\tilde{a}_{k_i}}}{1 - \alpha_{a_{k_i}} \alpha_{\tilde{a}_{k_i}}}, \end{aligned}$$

from which the desired result follows.

Finally, if $d > M$, then the sum $\tilde{\sigma}_d(N)$ is empty, and both sides of (16) are 0. ■

To estimate $\tilde{\sigma}_d(N)$, we use the following result for elementary symmetric functions.

Lemma 18. [3, p. 147, Lemma 13] *Let $y_1, y_2, \dots, y_{M'}$ be M' non-negative real numbers. For each positive integer d , not exceeding M' , let*

$$\sigma_d = \sum_{1 \leq k_1 < \dots < k_d \leq M'} y_{k_1} y_{k_2} \dots y_{k_d},$$

so that σ_d is the d -th elementary symmetric function of the y_k 's. Then, for each d , we have

$$(19) \quad \frac{1}{d!} \sigma_1^d \left(1 - \binom{d}{2} \frac{1}{\sigma_1^2} \sum_{k=1}^{M'} y_k^2 \right) \leq \sigma_d \leq \frac{1}{d!} \sigma_1^d,$$

where we interpret $\binom{d}{2}$ to be 0 when $d = 1$.

The next lemma gives us bounds on the probability of the event $\mathfrak{e}(N, d)$ in terms of λ_N and λ'_N .

Lemma 20. *Let N and d be non-negative integers. Then we have*

$$(21) \quad P(\mathfrak{e}(N, d)) \leq \frac{(\lambda'_N)^d}{d!} e^{-\lambda_N}.$$

Furthermore, if $d \leq M$, we have

$$(22) \quad P(\mathfrak{e}(N, d)) \geq \frac{(\lambda'_N)^d}{d!} e^{-\lambda'_N} \left(1 - \binom{d}{2} (\lambda'_N)^{-2} Q^* \right),$$

where

$$Q^* = \sum_{1 \leq k \leq M} \left(\frac{\alpha_{a_k} \alpha_{\tilde{a}_k}}{1 - \alpha_{a_k} \alpha_{\tilde{a}_k}} \right)^2,$$

and $\binom{d}{2}$ is interpreted to be 0 if $d < 2$.

PROOF. We note that if $d > M$, then the event $\mathfrak{e}(N, d)$ is empty and (21) is trivial. Suppose $1 \leq d \leq M$. We apply (19) with $M' = M$ and $y_k = \alpha_{a_k} \alpha_{\tilde{a}_k} / (1 - \alpha_{a_k} \alpha_{\tilde{a}_k})$ to estimate $\tilde{\sigma}_d(N)$ in (16); thus noting that $\tilde{\sigma}_1(N) = \lambda'_N$, we obtain

$$P(\mathfrak{e}(N, d)) \leq \left(\prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \right) \frac{(\lambda'_N)^d}{d!},$$

and

$$P(\mathfrak{e}(N, d)) \geq \left(\prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \right) \frac{(\lambda'_N)^d}{d!} \left(1 - \binom{d}{2} (\lambda'_N)^{-2} Q^* \right).$$

Applying the inequality $e^{-t/(1-t)} < 1 - t < e^{-t}$, which holds for $0 < t < 1$, with $t = \alpha_{a_k} \alpha_{\tilde{a}_k}$ ($1 \leq k \leq M$), we obtain

$$(23) \quad e^{-\lambda'_N} < \left(\prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}) \right) < e^{-\lambda_N},$$

and our result follows. When $d = 0$, we have

$$P(\mathfrak{e}(N, d)) = \prod_{1 \leq k \leq M} (1 - \alpha_{a_k} \alpha_{\tilde{a}_k}),$$

and the result is immediate from (23) in this case. ■

To estimate λ_N and λ'_N , we first prove the following lemma.

Lemma 24. *If Hypothesis A is satisfied, then*

$$(25) \quad \lambda'_N \sim \lambda_N$$

as $N \rightarrow \infty$.

PROOF. Recall from (11) that if $1 \leq k \leq M$, then $q^n \leq \tilde{a}_k < q^{n+1}$. Consequently, we have

$$\alpha_{a_k} \alpha_{\tilde{a}_k} < \alpha_{\tilde{a}_k} \leq \alpha_{q^n} = \mathbf{o}(1)$$

as $N \rightarrow \infty$. Therefore, we obtain

$$\begin{aligned} \lambda'_N - \lambda_N &= \sum_{1 \leq k \leq M} \alpha_{a_k} \alpha_{\tilde{a}_k} \left(\frac{1}{1 - \alpha_{a_k} \alpha_{\tilde{a}_k}} - 1 \right) \\ &= \sum_{1 \leq k \leq M} \alpha_{a_k} \alpha_{\tilde{a}_k} \left(\frac{\alpha_{a_k} \alpha_{\tilde{a}_k}}{1 - \alpha_{a_k} \alpha_{\tilde{a}_k}} \right) \\ &\leq \alpha_{q^n} \lambda'_N, \end{aligned}$$

from which the result follows. ■

Therefore, it is enough to estimate λ_N . The following lemma gives us an estimate for λ_N sufficient for our purpose.

Lemma 26. *Suppose that the sequence $\{\alpha_j\}$, introduced in Theorem 10, is such that*

$$\alpha_j = \alpha \frac{(\log j)^{c'}}{j^c}$$

for $j \geq j_0$; where j_0, α, c, c' are constants such that $\alpha > 0$, $0 < \alpha_j < 1$ ($j \geq 0$), $0 < c < 1$, and $c' \geq 0$. Then, for sufficiently large H , there exist positive constants D_1 and D_2 , which depend at most on c, c' , and q , such that

$$(27) \quad \alpha^2 D_1 (\log N)^{2c'} q^{n(1-2c)} < \lambda_N < \alpha^2 D_2 (\log N)^{2c'} q^{n(1-2c)}$$

for all $N > H$.

Furthermore, we have

$$(28) \quad m_N^* \sim \frac{\alpha}{1-c} (\log N)^{c'} N^{(1-c)}.$$

Finally, if $c' = 0$, then with probability 1, the numbers b_j of the sequence $\omega = \{p_{b_j}\}$ satisfy

$$(29) \quad b_j \sim \left(\frac{1-c}{\alpha} j \right)^{1/(1-c)}$$

as $j \rightarrow \infty$.

PROOF. We begin by finding a lower bound for λ_N . We assume $p \neq 2$. The case $p = 2$ can be treated in a similar manner. Suppose $N > q \cdot (j_0 + 1)$ from which it follows that $q^n > j_0$. Let C_0 and C'_0 be the positive constants defined by

$$C_0 = \sum_{1 \leq j < j_0} \alpha_j,$$

and

$$C'_0 = \sum_{1 \leq j < j_0} \frac{(\log j)^{c'}}{j^c}.$$

Since $q^n \leq \tilde{a}_i < q^{n+1}$, $0 \leq a_i < q^{n+1}$, and $(\log x)^{c'}/x^c$ is a decreasing function, we obtain

$$\begin{aligned} \lambda_N &= \sum_{1 \leq i \leq M} \alpha_{a_i} \alpha_{\tilde{a}_i} > \frac{\alpha (\log q^{n+1})^{c'}}{q^{(n+1)c}} \sum_{1 \leq i \leq M} \alpha_{a_i} \\ &> \frac{\alpha^2 (\log q^{n+1})^{c'}}{q^{(n+1)c}} \left(\sum_{j=(q^{n+1}-1)/2}^{q^{n+1}-1} \frac{(\log j)^{c'}}{j^c} - C'_0 \right) \\ &> \frac{\alpha^2 (\log q^n)^{2c'}}{q^{(n+1)c}} \left(\sum_{j=(q^{n+1}-1)/2}^{q^{n+1}-1} \frac{1}{j^c} - C'_0 \right). \end{aligned}$$

We know that for all $s, t \in \mathbb{N}$, $0 < s < t$,

$$(30) \quad \frac{1}{1-c} (t+1)^{1-c} - \frac{1}{1-c} s^{1-c} \leq \sum_{s \leq j \leq t} \frac{1}{j^c} \leq \frac{1}{1-c} t^{1-c} - \frac{1}{1-c} (s-1)^{1-c}.$$

Thus, by (30) we can give the following lower bound for λ_N ,

$$(31) \quad \begin{aligned} \lambda_N &> \frac{\alpha^2 (\log q^n)^{2c'}}{q^{(n+1)c}} \left(\frac{1}{1-c} (q^{n+1})^{1-c} - \frac{1}{1-c} \left(\frac{q^{n+1}-1}{2} \right)^{1-c} - C'_0 \right) \\ &= \frac{\alpha^2 (\log q^n)^{2c'}}{(1-c)q^c} q^{n(1-2c)} \left(q^{1-c} - \left(\frac{q}{2} - \frac{1}{2q^n} \right)^{1-c} - \frac{1-c}{q^{n(1-c)}} \cdot C'_0 \right). \end{aligned}$$

Since $q^n \leq N < q^{n+1}$, we have $\log N(1 - \log q / \log N) < \log q^n$. It follows from (31) that by taking H sufficiently large, we obtain

$$(32) \quad \lambda_N > \alpha^2 \frac{q^{1-2c}}{2(1-c)} \left(1 - \frac{1}{2^{1-c}} \right) (\log N)^{2c'} q^{n(1-2c)}$$

for all $N > H$.

Next, we would like to find an upper bound for λ_N . Again, since $q^n \leq \tilde{a}_i < q^{n+1}$ and $0 \leq a_i < q^{n+1}$, by similar calculations as before we have

$$\lambda_N < \frac{\alpha (\log q^{n+1})^{c'}}{q^{nc}} \sum_{1 \leq i \leq M} \alpha_{a_i} < \frac{\alpha (\log q^{n+1})^{c'}}{q^{nc}} \left(C_0 + \alpha \sum_{j=1}^M \frac{(\log q^{n+1})^{c'}}{j^c} \right).$$

Thus, by applying (30), we obtain

$$\begin{aligned} \lambda_N &< \frac{\alpha^2 (\log q^{n+1})^{2c'}}{q^{nc}} \left(\frac{C_0}{\alpha (\log q^{n+1})^{c'}} + \frac{1}{1-c} \left(\frac{q^{n+1}-1}{2} \right)^{1-c} \right) \\ &= \frac{\alpha^2 (\log q^{n+1})^{2c'}}{(1-c)} q^{n(1-2c)} \left(\frac{C_0(1-c)}{\alpha (\log q^{n+1})^{c'} q^{n(1-c)}} + \left(\frac{q}{2} - \frac{1}{2q^n} \right)^{1-c} \right). \end{aligned}$$

Therefore, by taking H sufficiently large, we obtain that

$$\lambda_N < \alpha^2 \frac{2^{2c'+1}}{(1-c)} \left(\frac{q}{2} \right)^{1-c} (\log q^n)^{2c'} q^{n(1-2c)}$$

for all $N > H$. Then, since $q^n \leq N < q^{n+1}$, we are done with the first part of the lemma.

Clearly, we have

$$m_N^* = \sum_{j=1}^N \alpha \frac{(\log j)^{c'}}{j^c} + \mathbf{O}(1) = (1 + \mathbf{o}(1)) \frac{\alpha}{1-c} (\log N)^{c'} N^{(1-c)},$$

and this proves (28). We note (28) shows that (13) and (14) are satisfied.

The final assertion of the lemma follows from (28), in view of Lemma 12, and the fact that $s_{b_j}^*(\omega) = j$ for $\omega = \{p_{b_j}\}_{j \in \mathbb{N}}$; for in this way it follows that, if $c' = 0$, we have with probability 1,

$$j = s_{b_j}^*(\omega) \sim m_{b_j}^* \sim \frac{\alpha}{1-c} b_j^{1-c},$$

or equivalently, $b_j \sim \left(\frac{1-c}{\alpha} j \right)^{1/(1-c)}$. ■

We also make use of the following lemma.

Lemma 33. [3, p. 149, Lemma 17] *If $0 < \xi \leq U$, then*

$$\sum_{d \geq U} \frac{\xi^d}{d!} \leq \left(\frac{e\xi}{U} \right)^U,$$

and if $0 < V \leq \xi$, then

$$\sum_{0 \leq d \leq V} \frac{\xi^d}{d!} \leq \left(\frac{e\xi}{V} \right)^V.$$

5. PROOF OF THEOREM 7

Let $c = c' = 1/2$. We choose a number $\alpha > 0$ to satisfy

$$\alpha^2 \frac{q^{1-2c}}{2(1-c)} \left(1 - \frac{1}{2^{1-c}} \right) > 1.$$

We then define a sequence $\{\alpha_j\}$ by

$$(34) \quad \alpha_j = \alpha \left(\frac{\log j}{j} \right)^{1/2}$$

for all $j \geq j_0$, where j_0 is a positive integer sufficient large such that the expression in (34) is less than $1/2$ for all $j \geq j_0$. For $1 \leq j < j_0$, we let $\alpha_j = 1/2$. The precise value of α_j for small j is unimportant, but the above choices ensure $0 < \alpha_j < 1$, so that Hypothesis A is satisfied. By (32) in the proof of Lemma 26, we have for all N sufficiently large that

$$(35) \quad \lambda_N \geq \alpha^2 D_1 \log N > \log N.$$

Hence, we know there exists $\delta > 0$ such that

$$(36) \quad e^{-\lambda_N} \ll N^{-1-\delta}.$$

We establish the theorem by showing that, with probability 1, $\log N \ll r_N(\omega) \ll \log N$ for large N , or equivalently (in view of Lemmas 24 and 26)

$$(37) \quad \lambda'_N \ll r_N(\omega) \ll \lambda'_N$$

for $N > N_0(\omega)$. We apply the Borel-Cantelli lemma twice to prove that each of the two assertions of (37) holds with probability 1. For this purpose, we must show that if C_1, C_2 are suitably chosen positive constants, then we have

$$(38) \quad \sum_{N=0}^{\infty} P(\{\omega \in \Omega: r_N(\omega) > C_1 \lambda'_N\}) < \infty$$

and

$$(39) \quad \sum_{N=0}^{\infty} P(\{\omega \in \Omega: r_N(\omega) < C_2 \lambda'_N\}) < \infty.$$

By Lemmas 20 and 33, we have

$$\begin{aligned} P(\{\omega \in \Omega: r_N(\omega) > C_1 \lambda'_N\}) &\leq e^{-\lambda_N} \sum_{d \geq C_1 \lambda'_N} \frac{(\lambda'_N)^d}{d!} \\ &\leq e^{-\lambda_N} \left(\frac{e}{C_1} \right)^{C_1 \lambda'_N}, \end{aligned}$$

provided $C_1 \geq 1$. Thus, by choosing $C_1 = e$, we obtain a bound $e^{-\lambda_N}$ for the summand of (38), and the inequality (38) follows from (36).

On the other hand, again by Lemmas 20 and 33, we obtain the following estimate for the summand of (39),

$$\begin{aligned} P(\{\omega \in \Omega: r_N(\omega) < C_2 \lambda'_N\}) &\leq e^{-\lambda_N} \sum_{0 \leq d \leq C_2 \lambda'_N} \frac{(\lambda'_N)^d}{d!} \\ &\leq e^{-\lambda_N} \left(\frac{e}{C_2} \right)^{C_2 \lambda'_N}, \end{aligned}$$

provided $C_2 \leq 1$. Thus, it suffices to show that C_2 can be chosen to satisfy, in addition to $0 < C_2 \leq 1$,

$$\left(\frac{e}{C_2} \right)^{C_2 \lambda'_N} \ll N^{\delta/2};$$

for (39) will then follow from (36). By Lemmas 24 and 26, we know there exists $D > 0$ such that $\lambda'_N \leq D \log N$ for N sufficiently large. Therefore, we only need to choose a small positive constant C_2 satisfying

$$\left(\frac{e}{C_2} \right)^{C_2} \leq e^{\delta/(2D)},$$

which is certainly possible since $(e/t)^t \rightarrow 1$ as $t \rightarrow 0$ from the positive side.

We have now shown that ω has each of the desired properties with probability 1, and this proves the theorem. ■

6. PROOF OF THEOREM 8

Let $\epsilon > 0$ be given. We define a sequence $\{\alpha_j\}$ by $\alpha_0 = 1/2$ and

$$\alpha_j = \frac{1}{2^{j^{1-1/(2+\epsilon)}}}$$

for $j \geq 1$. It then follows by Lemma 26 (with $\alpha = 1/2$, $c = 1 - 1/(2 + \epsilon)$, and $c' = 0$) that, with probability 1, $\omega = \{p_{b_j}\}$ satisfies $b_j \sim c^* j^{2+\epsilon}$, where c^* is some positive constant.

Since the sequence $\{\alpha_j\}$ satisfies Hypothesis A, we have $\lambda'_N \sim \lambda_N$ by Lemma 24. Thus, by Lemma 26 we know that there exist positive constants D_1 and D_2 such that

$$(40) \quad D_1 q^{-\epsilon n/(2+\epsilon)} < \lambda_N, \lambda'_N < D_2 q^{-\epsilon n/(2+\epsilon)}$$

for N sufficiently large.

We again appeal to the Borel-Cantelli Lemma. It follows from this lemma that if a positive number K satisfies the property

$$(41) \quad \sum_{N=0}^{\infty} P(\{\omega \in \Omega: r_N(\omega) \geq K\}) < \infty,$$

then, with probability 1, we have

$$r_N(\omega) < K$$

for $N > N_0(\omega)$.

We note that, by (40), $\lambda_N \rightarrow 0$ and $\lambda'_N \rightarrow 0$ as $N \rightarrow \infty$. Thus, by Lemmas 20 and 33, we obtain the following estimate for the summand of (41),

$$P(\{\omega \in \Omega: r_N(\omega) \geq K\}) \leq e^{-\lambda_N} \sum_{d \geq K} \frac{(\lambda'_N)^d}{d!} \leq e^{-\lambda_N} \left(\frac{e\lambda'_N}{K} \right)^K \ll (\lambda'_N)^K$$

for N sufficiently large. Since $q^n \leq N < q^{n+1}$, we have

$$(\lambda'_N)^K \leq D_2^K q^{-\epsilon n K / (2+\epsilon)} \ll N^{-\epsilon K / (2+\epsilon)}.$$

Therefore, provided $\epsilon K / (2 + \epsilon) > 1$, or equivalently,

$$K > 1 + 2\epsilon^{-1},$$

it is clear that (41) is achieved. Accordingly we have, with probability 1,

$$r_N(\omega) < 2(1 + \epsilon^{-1})$$

for $N > N_1(\epsilon, \omega)$. This completes the proof of the theorem. ■

7. PROOF OF THEOREM 9

Recall we defined $t_N(\omega)$ to be

$$t_N(\omega) = |\{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}: p_a, p_b \in \omega, p_N = p_a - p_b, \deg p_a, \deg p_b \leq \deg p_N\}|.$$

As before given $p_N \in \mathbb{F}_q[T]$, we let $n := n(N) = \deg p_N = \lfloor \log_q N \rfloor$. It is clear that for $p_N \neq 0$, there exist q^{n+1} pairs of polynomials (p_a, p_b) such that $p_N = p_a - p_b$ and $\deg p_a, \deg p_b \leq n$. Also, every polynomial of degree less than or equal to n will appear as p_a and p_b exactly once. Let $S_{\hat{u}, n}$ denote the set of all polynomials in $\mathbb{F}_q[T]$ whose degree are less than or equal to n , and the coefficient of T^n is $\hat{u} \in \mathbb{F}_q$. Clearly, we have $|S_{\hat{u}, n}| = q^n$. If we consider each polynomial in $S_{\hat{u}, n}$ as p_b , then the corresponding set of p_a 's is $S_{u, n}$ for some $u \neq \hat{u}$ as $\deg p_N = n$.

For each $u \in \mathbb{F}_q$, we consider

$$t_{N, u}(\omega) = |\{(a, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}: p_N = p_a - p_b, \text{ where } p_a, p_b \in \omega \text{ and } p_a \in S_{u, n}\}|.$$

If $p_N = p_a - p_b$, we relabel p_b as $p_{\hat{a}}$ to make its correspondence with p_a more explicit. We form the following two disjoint sets

$$\mathcal{A}_N = \{a \in \mathbb{Z}_{\geq 0}: p_a \in S_{u, n}\} = \{\iota^{-1}(u)q^n, \dots, (\iota^{-1}(u) + 1)q^n - 1\}$$

and

$$\hat{\mathcal{A}}_N = \{\hat{a} \in \mathbb{Z}_{\geq 0}: p_{\hat{a}} \in S_{\hat{u}, n}\} = \{\iota^{-1}(\hat{u})q^n, \dots, (\iota^{-1}(\hat{u}) + 1)q^n - 1\}.$$

Let $M_0 := M_0(N) = |\mathcal{A}_N| = |\hat{\mathcal{A}}_N| = q^n$. For convenience, we label the M_0 elements of \mathcal{A}_N by a_i ($1 \leq i \leq M_0$), and the corresponding elements of $\hat{\mathcal{A}}_N$ by \hat{a}_i , in other words we have $p_N = p_{a_i} - p_{\hat{a}_i}$ ($1 \leq i \leq M_0$).

We also define $\lambda_{N,u}$ and $\lambda'_{N,u}$ to be

$$\lambda_{N,u} = \sum_{1 \leq i \leq M_0} \alpha_{a_i} \alpha_{\widehat{a_i}},$$

and

$$\lambda'_{N,u} = \sum_{1 \leq i \leq M_0} \frac{\alpha_{a_i} \alpha_{\widehat{a_i}}}{1 - \alpha_{a_i} \alpha_{\widehat{a_i}}}.$$

With this set up we can recover analogues of all the previous lemmas in terms of M_0 , $\lambda_{N,u}$, $\lambda'_{N,u}$, and $t_{N,u}(\omega)$, in place of M , λ_N , λ'_N , and $r_N(\omega)$, respectively. Therefore, by a similar argument we obtain Theorem 8 with $t_{N,u}(\omega)$ in place of $r_N(\omega)$. Since this result holds with probability 1, and

$$t_N(\omega) = \sum_{u \in \mathbb{F}_q} t_{N,u}(\omega),$$

we have our result. ■

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